

Twofold Translative Tilings with Convex Bodies

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Abstract

Let K be a convex body. It is known that, in general, if K is a k -fold translative tile (for some positive integer k), then K may not be a (onefold) translative tile. However, in this paper I will show that for every convex body K , K is a twofold translative tile if and only if K is a translative tile.

Keywords Multiple tiling · Twofold tiling · Convex body · Polytope · Translative tile · Lattice tile

Mathematics Subject Classification 52C20 · 52C22

1 Introduction

Let D be a connected subset of \mathbb{R}^n , and let k be a positive integer. We say that a family of convex bodies $\{K_1, K_2, \dots\}$ is a *k-fold tiling of D* , if each point of D which does not lie in the boundary of any K_i , belongs to exactly k convex bodies of the family.

Let K be an n -dimensional convex body, and let X be a discrete multisubset of \mathbb{R}^n . Denote by $K + X$ the family

$$\{K + x : x \in X\},$$

where $K + x = \{y + x : y \in K\}$. We say that the family $K + X$ is a *k-fold translative tiling with K* , if $K + X$ is a k -fold tiling of \mathbb{R}^n . In particular, if $X = \Lambda$ is a lattice, then $K + \Lambda$ is called a *k-fold lattice tiling with K* . We call K a *k-fold translative (lattice) tile* if there exists a k -fold translative (lattice) tiling with K . A onefold tiling (tile) is simply called a tiling (tile).

Let P be an n -dimensional centrally symmetric polytope with centrally symmetric facets. A *belt* of P is the collection of its facets which contain a translate of a given subfacet $((n - 2)$ -face) of P .

For the case of onefold tilings, Venkov [1] and McMullen [2] proved the following result.

Theorem 1.1. *A convex body K is a translative tile if and only if K is a centrally symmetric polytope with centrally symmetric facets, such that each belt of K contains four or six facets.*

Furthermore, a consequence of the proof of this result is that, every convex translative tile is also a lattice tile. In the case of general k -fold tilings, Gravin, Robins and Shiryaev [3] proved that

Theorem 1.2. *If a convex body K is a k -fold translative tile, for some positive integer k , then K is a centrally symmetric polytope with centrally symmetric facets.*

Moreover, they also showed that, every rational polytope P that is centrally symmetric and has centrally symmetric facets must be a k -fold lattice tile, for some positive integer k . This result implies that there exists a polytope P such that P is a k -fold translative tile (for some $k > 1$), but P is not a translative tile. For example, the octagon shown in Fig. 1 is a 7-fold lattice tile, but is not a translative tile.

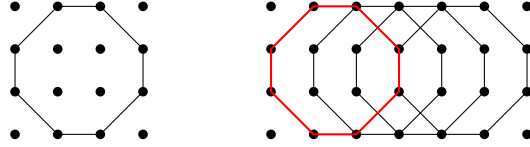


Fig. 1: The octagon that is a 7-fold lattice tile

In this paper, I will prove the following surprising result:

Theorem 1.3. *A convex body K is a twofold translative tile if and only if K is a onefold translative tile.*

In order to prove this result, I will modify the method used in [2]. As an immediate consequence of Theorem 1.3, we have

Corollary 1.4. *A convex body K is a twofold translative tile if and only if K is a twofold lattice tile.*

2 Some Notations

Let K be an n -dimensional convex body, and let X be a discrete multisubset of \mathbb{R}^n which contains the origin. Let q be a point on the boundary ∂K of K . We define

$$X_K(q) = \{x \in X : q \in K + x\},$$

$$\overset{\circ}{X}_K(q) = \{x \in X_K(q) : q \in \text{int}(K + x)\},$$

and

$$\partial X_K(q) = \{x \in X_K(q) : q \in \partial(K + x)\}.$$

In addition, we define

$$\overline{\partial} X_K(q) = \{x \in \partial X_K(q) : \text{int}(K) \cap (K + x) = \emptyset\},$$

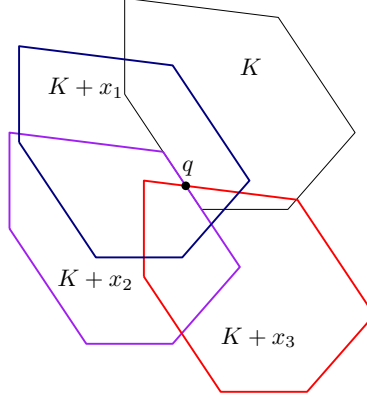


Fig. 2: $X_K(q) = \{0, x_1, x_2, x_3\}$, $\overset{\circ}{X}_K(q) = \{x_1\}$, $\partial X_K(q) = \{0, x_2, x_3\}$ and $\overline{\partial} X_K(q) = \{x_2\}$

(see Fig. 2 for an example).

Now suppose that P is an n -dimensional centrally symmetric convex polytope with centrally symmetric facets. Let G be a translate of a subfacet of P . Denote by $\mathcal{B}_P(G)$ the belt of P determined by G . Let q be a point that lies in a facet in $\mathcal{B}_P(G)$. Let $S(G, q)$ be the $(n-2)$ -dimensional plane that contains the point q and parallels to G . We define

$$\dot{\partial} X_P(G, q) = \{x \in \overline{\partial} X_P(q) : P \cap (P+x) \subset S(G, q)\}.$$

Let F be a subset of ∂P containing the point q , we define

$$\overline{\partial} X_P(G, F, q) = \{x \in \overline{\partial} X_P(q) : S(G, q) \cap F \cap (P+x) \neq F \cap (P+x)\},$$

(see Fig. 3).

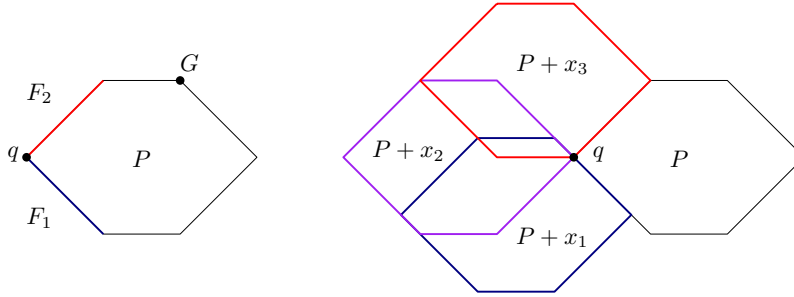


Fig. 3: $\dot{\partial} X_P(G, q) = \{x_2\}$, $\overline{\partial} X_P(G, F_1, q) = \{x_1\}$ and $\overline{\partial} X_P(G, F_2, q) = \{x_3\}$

Let $F_P(G, q)$ be the union of those facets in $\mathcal{B}_P(G)$ which contain q . It is easy to see that, $S(G, q)$ divides $F_P(G, q)$ into two parts. After choosing a

direction, we may define these two parts $F_P^+(G, q)$ and $F_P^-(G, q)$ as shown in Fig. 4. Denote by $\text{Ang}_P(G, q)$ the angle from $F_P^+(G, q)$ to $F_P^-(G, q)$, and denote by $\text{ang}_P(G, q)$ the measure of $\text{Ang}_P(G, q)$ in radian. Obviously, if q lies in some subfacet that parallels to G , then $\text{ang}_P(G, q) < \pi$, otherwise, $\text{ang}_P(G, q) = \pi$. We will denote by $E_P^+(G, q)$ the subfacet which parallels to G and is contained in $F_P^+(G, q)$, but is not containing q (Fig. 4). We can also define $E_P^-(G, q)$ in the similar way.

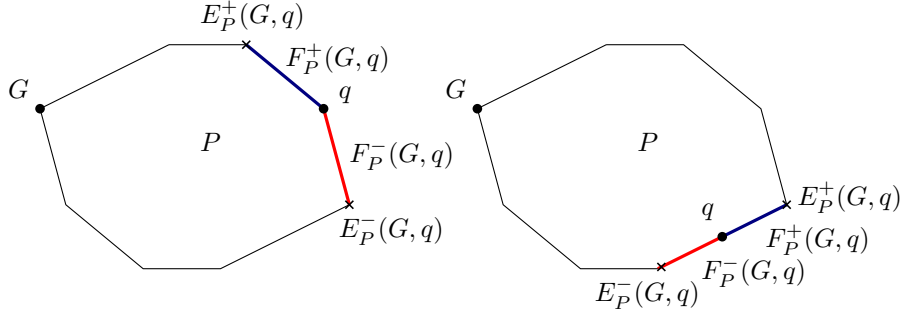


Fig. 4: $F_P^+(G, q)$, $F_P^-(G, q)$, $E_P^+(G, q)$ and $E_P^-(G, q)$

3 Some Lemmas

For a positive real number ε and a point p , denote by $B_\varepsilon(p)$ the closed ball with center p and radius ε .

Lemma 3.1. *Suppose that K and K' are convex bodies. If there exist a point $p \in \partial K \cap \partial K'$ and a positive real number ε such that $B_\varepsilon(p) \cap \text{int}(K) \cap \text{int}(K') = \emptyset$, then $\text{int}(K) \cap \text{int}(K') = \emptyset$.*

Proof. Since $B_\varepsilon(p) \cap K$ and $B_\varepsilon(p) \cap K'$ are convex, by applying the basic result of Convex and Discrete Geometry, we know that there is a hyperplane H which separates $B_\varepsilon(p) \cap K$ and $B_\varepsilon(p) \cap K'$. Assume that $p' \in \text{int}(K) \cap \text{int}(K')$. By the convexity, the line segment L between the point p and the point p' must lie in $K \cap K'$. Therefore, $L \cap B_\varepsilon(p)$ must be contained in the hyperplane H , and hence $p' \in H$. On the other hand, there is a positive real number δ such that $B_\delta(p') \subset \text{int}(K) \cap \text{int}(K')$. So $B_\delta(p') \subset H$, this is impossible. \square

Lemma 3.2. *Let D be a connected subset of \mathbb{R}^n , and let k be a positive integer. Suppose that a family of convex bodies $\{K_1, K_2, \dots\}$ is a k -fold tiling of D . We have that, for every $i \in \{1, 2, \dots\}$ and every point $q \in \partial K_i$, if q is an interior point of D , then there must be a $j \in \{1, 2, \dots\}$ such that $q \in \partial K_j$ and $\text{int}(K_i) \cap \text{int}(K_j) = \emptyset$.*

Proof. For $i \neq j$, let

$$A_i^j = \{p \in \partial K_i : B_\varepsilon(p) \cap \partial K_j \cap \text{int}(K_i) = \emptyset, \text{ for some } \varepsilon > 0\},$$

and

$$B_i^j = \{p \in \partial K_i \setminus A_i^j : B_\varepsilon(p) \cap \partial K_j \subset K_i, \text{ for some } \varepsilon > 0\}.$$

We note that, if $p \in \partial K_i \setminus (A_i^j \cup B_i^j)$, then p must lie in ∂K_j and for all $\varepsilon > 0$, we have $B_\varepsilon(p) \cap \partial K_j \cap \text{int}(K_i) \neq \emptyset$ and $(B_\varepsilon(p) \cap \partial K_j) \setminus K_i \neq \emptyset$ (Fig. 5). Obviously, when $K_i \cap K_j = \emptyset$, we have $A_i^j = \partial K_i$ and $B_i^j = \emptyset$. We note that for a fixed i , there are finitely many j such that $K_i \cap K_j \neq \emptyset$. Hence, there are finitely many j such that $A_i^j \cup B_i^j \neq \partial K_i$. Let

$$C_i = \bigcap_{j \neq i} A_i^j \cup B_i^j.$$

By considering $(n-1)$ -dimensional Lebesgue measure, one can show that the closure of C_i is ∂K_i . Therefore, to prove the lemma, it suffices to show that the statement is true for all $q \in C_i$.

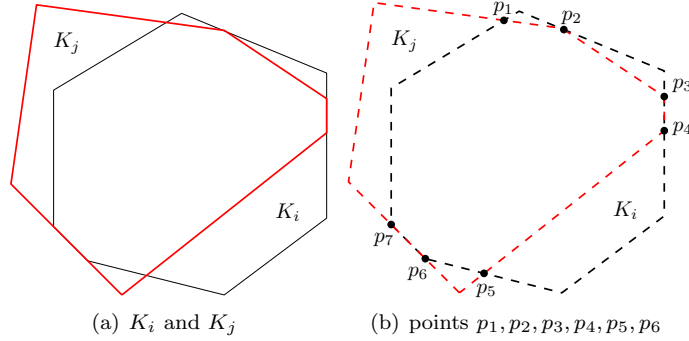


Fig. 5: $A_i^j = \partial K_i \setminus \{p_1, p_2, p_3, p_4, p_5\}$, $B_i^j = \{p_2, p_3, p_4\}$ and $\partial K_i \setminus (A_i^j \cup B_i^j) = \{p_1, p_5\}$

Suppose that $q \in C_i \cap \text{int}(D)$. Denote by $\mathcal{F}(q)$ the collection of convex bodies K_j containing the point q . We note that $\mathcal{F}(q)$ must be finite. Let

$$\mathcal{F}'(q) = \{K_1, K_2, \dots\} \setminus \mathcal{F}(q).$$

It is not hard to see that, there exists a positive real number ε_0 such that $B_{\varepsilon_0}(q) \cap K_j = \emptyset$, for any $K_j \in \mathcal{F}'(q)$. Since $q \in C_i \cap \text{int}(D)$, we may assume, without loss of generality, that $B_{\varepsilon_0}(q) \subset \text{int}(D)$ and for each j , we have $B_{\varepsilon_0}(q) \cap \partial K_j \cap \text{int}(K_i) = \emptyset$ or $B_{\varepsilon_0}(q) \cap \partial K_j \subset K_i$. Furthermore, we may also assume that for all j , both $B_{\varepsilon_0}(q) \cap K_j$ and $B_{\varepsilon_0}(q) \setminus \text{int}(K_j)$ are connected. For a unit vector u , we denote by $R(q, u)$ the ray parallel to u and starting at q . One can find a unit vector u that satisfies

- (i) $R(q, u) \cap K_i = \{q\}$ and $R(q, -u) \cap \text{int}(K_i) \neq \emptyset$,
- (ii) there is a point $q' \in R(q, u) \cap B_{\varepsilon_0}(q)$ such that $q' \notin \partial K_j$ for all $j = 1, 2, \dots$

Since $\{K_1, K_2, \dots\}$ is a k -fold tiling of D and $q' \in B_{\varepsilon_0}(q) \subset \text{int}(D)$, there exist exactly k convex bodies K_{i_1}, \dots, K_{i_k} such that $q' \in \text{int}(K_{i_j})$, $j = 1, \dots, k$. We note that $\{K_{i_1}, \dots, K_{i_k}\} \subset \mathcal{F}(q)$ and $K_i \notin \{K_{i_1}, \dots, K_{i_k}\}$. Denote by $\tilde{\mathcal{F}}(q)$ the collection of convex bodies K_j which contain the point q as an interior point. If $\{K_{i_1}, \dots, K_{i_k}\} \subset \tilde{\mathcal{F}}(q)$, then it is easy to see that $K_{i_1} \cap \dots \cap K_{i_k} \cap K_i$ must have an interior point, which is impossible, since $\{K_1, K_2, \dots\}$ is a k -fold tiling. Now we suppose that $K_{i_{j_0}} \notin \tilde{\mathcal{F}}(q)$, for some $j_0 \in \{1, \dots, k\}$. It is clear that $q \in \partial K_{i_{j_0}}$. We will show that $\text{int}(K_i) \cap \text{int}(K_{i_{j_0}}) = \emptyset$. Since $q \in C_i$, we have that $q \in A_i^{j_0}$ or $q \in B_i^{j_0}$. Recall that for each j , we have $B_{\varepsilon_0}(q) \cap \partial K_j \cap \text{int}(K_i) = \emptyset$ or $B_{\varepsilon_0}(q) \cap \partial K_j \subset K_i$. If $q \in B_i^{j_0}$, then $B_{\varepsilon_0}(q) \cap \partial K_{j_0} \subset K_i$. From this, one can deduce that $B_{\varepsilon_0}(q) \cap K_{i_{j_0}} \subset K_i$ which is impossible, since q' is in $B_{\varepsilon_0}(q) \cap K_{i_{j_0}}$ but is not in K_i . Therefore $q \in A_i^{j_0}$. It is not hard to see that $\partial K_{i_{j_0}}$ divides $B_{\varepsilon_0}(q)$ into two parts, where one of them does not contain any point of $\text{int}(K_i)$, we denote this part by B' . Since $q' \in \text{int}(K_{i_{j_0}})$, it is obvious that q' and $B_{\varepsilon_0}(q) \cap K_{i_{j_0}}$ must be contained in the same part. Because $q' \in R(q, u)$, by the property (i) of the vector u , we see that q' must lie in B' . Therefore $B_{\varepsilon_0}(q) \cap K_{i_{j_0}}$ is contained in B' , and hence $B_{\varepsilon_0}(q) \cap K_{i_{j_0}} \cap \text{int}(K_i) = \emptyset$. By Lemma 3.1, we obtain $\text{int}(K_i) \cap \text{int}(K_{i_{j_0}}) = \emptyset$. This completes the proof. \square

Corollary 3.3. *Let X be a discrete multisubset of \mathbb{R}^n containing the origin, and let k be a positive integer. Suppose that P is a centrally symmetric convex polytope with centrally symmetric facets, and G is its subfacet. Let q be a point on a facet in $\mathcal{B}_P(G)$. If $P + X$ is a k -fold tiling, then $\bar{\partial}X_P(G, F_P^+(G, q), q)$ and $\bar{\partial}X_P(G, F_P^-(G, q), q)$ are not empty.*

We will denote by $X_P^+(G, q)$ and $X_P^-(G, q)$ the sets $\bar{\partial}X_P(G, F_P^+(G, q), q)$ and $\bar{\partial}X_P(G, F_P^-(G, q), q)$, respectively. For example, in Fig. 3, we have $X_P^+(G, q) = \{x_1\}$ and $X_P^-(G, q) = \{x_3\}$.

4 Proof of Main Theorem

Lemma 4.1. *If a convex body K is a twofold translative tile, then K is a centrally symmetric polytope with centrally symmetric facets, such that each belt of K contains four or six facets.*

Proof. By Theorem 1.2, we know that K is a centrally symmetric polytope with centrally symmetric facets.

Let G be an arbitrary subfacet of K . Recall that we denote by $\mathcal{B}_K(G)$ the belt of K determined by G . Let $\mathcal{B}_K(G)$ have m pairs of opposite facets. We will show that $m \leq 3$. To do this, we shall suppose that $m \geq 4$, and obtain a contradiction. Denote by $K(G)$ the union of the facets which are not contained in $\mathcal{B}_K(G)$.

Suppose that $K + X$ is a twofold translative tiling, where X is a multisubset of \mathbb{R}^n . Without loss of generality, we may assume that $0 \in X$. It is not hard to see that, we can choose a point $q \in G$ to lie in none of $K(G) + x$, where $x \in X$.

First, we will show that $\overset{\circ}{X}_K(q) = \emptyset$. If there is a $x \in \overset{\circ}{X}_K(q)$, then by Corollary 3.3, there exist $x_1 \in X_K^+(G, q)$ and $x_2 \in X_K^-(G, q)$. Obviously, we have $x_1 \neq x_2$. Since $m \geq 4$, it is easy to see that both $\text{ang}_K(G, q) + \text{ang}_{K+x_1}(G, q)$ and $\text{ang}_K(G, q) + \text{ang}_{K+x_2}(G, q)$ are greater than $(m-1)\pi - (m-2)\pi = \pi$. Therefore $\text{Ang}_{K+x_1}(G, q)$ and $\text{Ang}_{K+x_2}(G, q)$ are not opposite angles, and hence $\text{ang}_K(G, q) + \text{ang}_{K+x_1}(G, q) + \text{ang}_{K+x_2}(G, q)$ is greater than $(m-1)\pi + (m-3)\pi = 2\pi$ (Fig. 6). This can be deduced that $(K+x) \cap (K+x_1) \cap (K+x_2)$ has an interior point which is impossible, since $K+X$ is a twofold tiling.

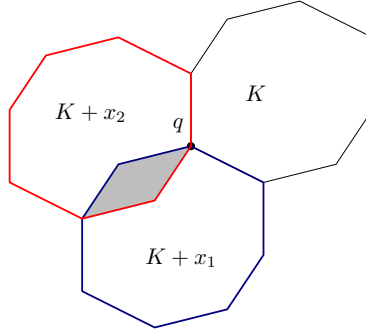


Fig. 6: $\text{ang}_K(G, q) + \text{ang}_{K+x_1}(G, q) + \text{ang}_{K+x_2}(G, q) > 2\pi$

We assert that $\overset{\circ}{X}_K(G, q) = \emptyset$. Suppose that $x \in \overset{\circ}{X}_K(G, q)$. Since K is centrally symmetric, it is not hard to see that $\text{Ang}_K(G, q)$ and $\text{Ang}_{K+x}(G, q)$ are opposite angles (Fig. 7). By Corollary 3.3, we can choose $x_1 \in X_K^+(G, q)$ and $x_2 \in X_K^-(G, q)$. Similar to the above argument, one obtains that $(K+x) \cap (K+x_1) \cap (K+x_2)$ has an interior point which is a contradiction.

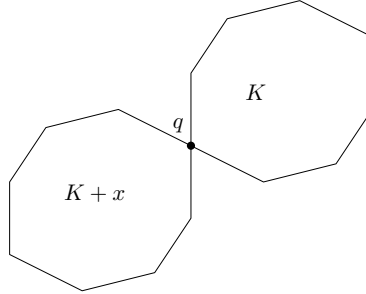


Fig. 7: $\text{Ang}_K(G, q)$ and $\text{Ang}_{K+x}(G, q)$

Now we will show that X must be a usual set (not a multiset). If not, then we may assume that 0 has multiplicity 2. By Corollary 3.3, one can choose $x_1 \in X_K^+(G, q)$ and $x_2 \in X_{K+x_1}^+(G, q)$ (see Fig. 8). From the above discussion, we know that $\text{Ang}_K(G, q)$, $\text{Ang}_{K+x_1}(G, q)$ and $\text{Ang}_{K+x_2}(G, q)$ cannot be opposite angles, hence $\text{ang}_K(G, q) + \text{ang}_{K+x_1}(G, q) + \text{ang}_{K+x_2}(G, q)$ is greater than 2π .

This implies that $K \cap (K + x_2)$ has an interior point. We note that $x_2 \neq 0$, and hence we obtain a contradiction.

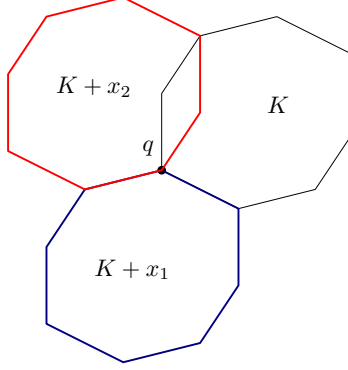


Fig. 8: $x_1 \in X_K^+(G, q)$ and $x_2 \in X_{K+x_1}^+(G, q)$

We shall divide the remaining proof into the following two cases:

- (i) *Case $m \geq 6$:* Since $\overset{\circ}{X}_K(q) = \emptyset$, we obtain $X_K(q) = \partial X_K(q)$. We note that $\text{ang}_K(G, q) < \pi$ and for each $x \in \partial X_K(q) \setminus \{0\}$, we have $\text{ang}_{K+x}(G, q) \leq \pi$. Because $K + X$ is a twofold tiling, so the cardinality of $\partial X_K(q)$ must be greater than 4. On the other hand, since $m \geq 6$, we know that the sum of five (distinct) non-opposite angles is greater than $(m-1)\pi - (m-5)\pi = 4\pi$. Therefore, the cardinality of $\partial X_K(q)$ cannot be greater than 4, this is a contradiction.
- (ii) *Case $m = 4$ or 5 :* Similar to the above, we have that the cardinality of $\partial X_K(q)$ is greater than 4. If there are two points $x, x' \in \partial X_K(q)$ such that q lies in the relative interior of a facet of $K + x$ and also lies in the relative interior of a facet of $K + x'$, then $\text{ang}_{K+x}(G, q) = \text{ang}_{K+x'}(G, q) = \pi$, and hence $\sum_{z \in \partial X_K(q)} \text{ang}_{K+z}(G, q)$ is greater than $\text{ang}_{K+x}(G, q) + \text{ang}_{K+x'}(G, q) + (m-1)\pi - (m-3)\pi = 4\pi$, which is impossible. Therefore, there is at most one point $x \in \partial X_K(q)$ such that p lies in the relative interior of a facet of $K + x$. We choose $x_1 \in X_K^+(G, q)$, $x_2 \in X_{K+x_1}^+(G, q)$ and $x_3 \in X_{K+x_2}^+(G, q)$. We note that $\text{ang}_{K+x_1} + \text{ang}_{K+x_2} < 2\pi$ and $\text{ang}_{K+x_1}(G, q) + \text{ang}_{K+x_2}(G, q) + \text{ang}_{K+x_3}(G, q) > 2\pi$. Hence, one can prove that $E_K^+(G, q) \cap \text{int}(K + x_3) \neq \emptyset$ (Fig. 9). Now we choose $q' \in E_K^+(G, q) \cap \text{int}(K + x_3)$ to lie in none of $K(G) + x$, where $x \in X$. Obviously, $x_3 \in \overset{\circ}{X}_K(q')$. On the other hand, by using the same argument as the proof of $\overset{\circ}{X}_K(q) = \emptyset$, one obtains $\overset{\circ}{X}_K(q') = \emptyset$. This is a contradiction.

Above all, we obtain $m \leq 3$. □

By Lemma 4.1 and Theorem 1.1, one obtain Theorem 1.3.

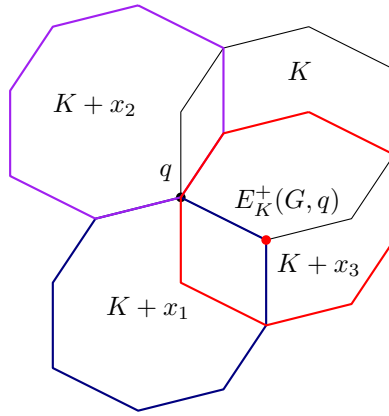


Fig. 9: $E_K^+(G, q) \cap \text{int}(K + x_3) \neq \emptyset$

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